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Connes' Distance Function for Commutative and Noncommutative Graphs

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By formulating the concept of a graph algebraically, i.e., as properties of the algebra of functions over the set of vertices and the set of edges, we arrive at a purely algebraic concept of distance related to the one proposed by Connes for manifolds which easily extends also to the noncommutative case. The Dirac operator used in Connes' approach is replaced by a generalized difference operator which can be defined on arbitrary graphs. We speculate on the question of how this operator might be related to the concept of a Dirac operator on graphs.

1. INTRODUCTION

Ever since the work of Connes on noncommutative geometry (Connes, 1994) many mathematical structures have been reformulated algebraically and their noncommutative counterparts examined. One of the simplest mathematical structures beyond that of a pure set is the notion of a graph. Special cases include lattices and clusters, which are frequently used in physics to replace continuous space or spacetime. It therefore seems natural to quantize also graphs along the ideas of Connes.

Despite the simplicity of the concept of a graph, it already implies some interesting structures, among them the notion of a distance $d(k, l)$ between two vertices k and l. This distance is defined as the length of the shortest path in the graph connecting the two vertices.

Recently, there has been some interest in reformulating this distance function on a graph using the notion of distance as defined by Connes for manifolds (Bimonte *et al.*, 1994; Atzmon, 1996; Dimakis and Müller-Hoissen,

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1998). Connes defines the geodesic distance between two states ω_1 and ω_2 over an algebra $\mathcal{F}$, characterizing a topological manifold, as

$$d(\omega_1, \omega_2) = \sup\{|\omega_1(f) - \omega_2(f)|; \text{ for } f \in \mathcal{F} \text{ with } \|[D, f]\| \leq 1\} \quad (1)$$

where D denotes a Dirac operator acting on $\mathcal{F}$ [or rather an unbounded, self-adjoint operator such that $(1 + D)^{-1}$ is compact]. We shall show in the following that this definition of a geodesic distance has a natural analog for graphs.

A first step toward an algebraic formulation of distance between two states on a graph involving functions $f \in \mathcal{F}_V$, where $\mathcal{F}_V$ is the (commutative) algebra of complex functions on the set of vertices, is the following:

$$d(\omega_1, \omega_2); \text{ eq } \sup\{|\omega_1(f) - \omega_2(f)|; f \in \mathcal{F}_V, \text{ such that } |f(p) - f(q)| \leq 1 \text{ if } p \text{ and } q \text{ are nearest neigbors}\} \quad (2)$$

For pure states, i.e, states ω for which $\omega(f) = f(k)$ for some vertex k, this definition is obviously equivalent to the length of the shortest path. However, the condition $|f(p) - f(q)| \leq 1$ if p and q are nearest neigbors makes explicit reference to neighbored vertices of the graph. If the notion of distance admits a noncommutative analog, this condition should be reformulated algebraically, i.e., by using appropriate operators acting on the function spaces over a graph. In Connes' approach the Dirac operator is used to define bounds on the slope of a function f on a manifold. However, for graphs there exist more natural structures to formulate this condition algebraically. This will be elaborated in this article.

In Section 2, we list the definition of a graph which seems most appropriate for an algebraic formulation. In Section 3 the algebraic formulation of a graph is given as well as the counterpart of Connes' distance function for arbitrary graphs. In Section 4 we briefly sketch how a graph may be reobtained from its algebra. Some comments on the operator D involved in this definition and its possible relation to the notion of a Dirac operator on graphs are contained in Section 5.

2. GRAPHS

The following definition of a graph as well as certain properties of graphs seem most appropriate for an algebraic formulation and a noncommutative extension.

A (general directed) graph consists of a set of vertices V, a set of edges E, and two mappings $\partial^{\pm}: E \rightarrow V$ (see, e.g., *Encyclopaedic Dictionary of Mathematics*, 1986). The image of an edge $e \in E$ under ∂^{+} (∂^{-}) will be called the final (initial) end vertex of e. We will always assume the sets V

and E to be finite or countable. One may impose further constraints on the mappings $\partial^{\pm}$ by the requirements

$$\partial^{+}(e) \neq \partial^{-}(e), \qquad \forall e \in E$$

and

$$\{\partial^{+}(e_1), \partial^{-}(e_1)\}, \neq \{\partial^{+}(e_2), \partial^{-}(e_2)\} \qquad \text{for any two } e_1 \neq e_2 \in E$$

In this way one arrives at so-called simple graphs, i.e., graphs without loops and multiple connections between two vertices. These constraints, however, will not be necessary for the following construction.

The above relations define a directed or oriented graph, i.e., each edge comes with a natural direction. In general, one may obtain undirected graphs from directed graphs by defining an equivalence relation. This will not be done here. Instead, we define an undirected simple graph to be a set of vertices V together with a symmetric, nonreflexive relation $E \subset V \times V$. Such a graph may be represented by its adjacency matrix, which for $k, l \in V$ is defined by

$$A_{kl} = \begin{cases} 1 & \text{if} \quad (k, l) \in E \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

A path on a graph connecting two vertices v and v' is defined to be a sequence of vertices $\{k_0 = v, k_1, k_2, \ldots, k_N = v'\}$ such that $(k_{i-1}, k_i) \in E$ for all $i = 1, \ldots, N$. Here N is called the length of the path. The distance $d(k, l)$ between two vertices k and l is equal to the length of the shortest path connecting k and l.

3. ALGEBRAIC FORMULATION OF A GRAPH AND THE DISTANCE FUNCTION

Let us now formulate the notion of a graph in terms of the function spaces over V and E.

Let $\mathcal{F}_V$ and $\mathcal{F}_E$ be the spaces of complex-valued functions over V and E, respectively. Both function spaces form a commutative C^*-algebra with respect to pointwise (or edgewise) multiplication, complex conjugation, and the supremum norm. The two mappings $\partial^{\pm}$ now induce two mappings $D^{\pm}$: $\mathcal{F}_V \rightarrow \mathcal{F}_E$ by

$$(D^{\pm}f)(e) = f(\partial^{\pm}e) \tag{4}$$

for any $f \in \mathcal{F}_V$ and any $e \in E$.

We also define the mapping

$$\mathrm{D}: \quad \mathscr{F}_V \to \mathscr{F}_E \quad \text{with} \quad \mathrm{D} = \mathrm{D}^+ - \mathrm{D}^- \tag{5}$$

This operator, which represents a kind of difference operator, will replace the Dirac operator in Connes' notion of distance. In the final section we will make some comments on how this operator may be related to the notion of a Dirac operator on graphs or lattices.

Connes' distance functional defines a distance for states, where a state ω is a linear, positive, normed functional on $\mathscr{F}_V$, i.e., for any two $f, g \in \mathscr{F}_V$ and $\alpha, \beta \in \mathbf{C}$ we have

$$\omega(\alpha f + \beta g) = \alpha\omega(f) + \beta\omega(g)$$

$$\omega(f^*f) \geq 0$$

$$\|\omega\| = 1, \quad \text{where} \quad \|\omega\| = \sup_{\|f\|=1}\{|\omega(f)|\}$$

(We may also define states over $\mathscr{F}_E$, but this is not required for our construction. See, however, the remarks in the final section.) States form a convex set, i.e., for any two states ω_1 and ω_2 also $\omega = \alpha\omega_1 + (1 - \alpha)\omega_2 (0 \leq \alpha \leq 1)$ is a state. Pure states are those states which cannot be represented as a nontrivial combination of other states. The pure states are in one-to-one correspondance with the set V, i.e., for each $v \in V$ there is a pure state ω_v with

$$\omega_v(f) = f(v) \tag{6}$$

Having defined the preliminaries, we now formulate the notion of a distance between two states. Let V be the set of vertices of an undirected graph and A be its adjacency matrix. Asign an arbitrary direction to each line, making the graph a directed graph with mappings $\partial^{\pm}$. Although the definition of distance requires the mappings $\mathrm{D}^{\pm}$ and therefore a directed graph, it will be obvious that the distance does not depend on the chosen directions and hence is a property of an undirected graph.

For any two states ω_1 and ω_2 we define

$$d(\omega_1, \omega_2) = \sup\{|\omega_1(f) - \omega_2(f)|; \text{ for } f \in \mathscr{F} \text{ with } \|\mathrm{D}f\| \leq 1\} \tag{7}$$

to be the distance between these two states. Since $\mathrm{D}f(e) = f(\partial^+(e)) - f(\partial^-(e))$, the condition $\|\mathrm{D} \leq 1\|$ implies that the maximum is taken with respect to all functions which differ between neighbored points in absolute value by not more than 1.

As the definition of $d(\omega_1, \omega_2)$ depends only on the absolute value $\|\mathrm{D}f\|$, this definition of distance does not depend on the choice of directions, i.e., it is well defined for an undirected graph.

Furthermore, for a graph with several components and ω_1 and ω_2 having their support on different components, the above definition of distance yields infinity, in accordance with the usual convention.

A similar distance functional has also been mentioned by Dimakis and Müller-Hoissen, 1998 (with reference to Dimakis and Müller-Hoissen, 1994) by defining a first-order differential calculus for the set of vertices of a graph. Our proposal here, however, makes the set of vertices *and* the set of edges of a graph the starting point. The formulation above only refers to the graph algebra—i.e., there is no explicit reference to the vertices or lines of the graph—so that the extension to noncommutative graph algebras (see below) is straightforward. Furthermore, in the final section we will comment on the possibility of relating D to a self-adjoint operator with the possible interpretation of a Dirac operator on a graph.

4. RECONSTRUCTION OF A GRAPH FROM ITS ALGEBRA

In the previous sections we summarized the definition of a graph and how an algebraic formulation may be derived. In this section we briefly sketch the reverse way. We start from an algebraic definition and derive the graph. This concept then may be generelized to a noncommutative formulation of graphs.

Algebraically, a (general directed) graph may be defined as two commutative C^*-algebras $\mathcal{V}$ and ε with identity elements id_V and id_E, respectively, together with two linear mappings $\mathrm{D}^{\pm}: \mathcal{V} \rightarrow \varepsilon$ satisfying the following properties:

$$\mathrm{D}^{\pm}(fg) = \mathrm{D}^{\pm}(f)\mathrm{D}^{\pm}(g) \tag{8}$$

$$\mathrm{D}^{\pm}(\mathrm{id}_V) = \mathrm{id}_E \tag{9}$$

$$\mathrm{D}^{\pm}(f^*) = \mathrm{D}^{\pm}(f)^* \tag{10}$$

These conditions are easily verified for the mappings $\mathrm{D}^{\pm}: \mathcal{F}_V \rightarrow \mathcal{F}_E$ defined in the previous section [Eq. (4)]. Note that the second condition does not necessarily follow from the first one, as $\mathrm{D}^{\pm}$ have not been required to be "onto." Furthermore, we have not required the mappings $\mathrm{D}^{\pm}$ to be C^*-algebra homomorphisms, as in general the norm is not preserved.

We now show how to reconstruct a graph from these conditions. Let V and E be the set of pure states in the dual spaces $\mathcal{V}^*$ and ε^*, respectively. The mappings $\mathrm{D}^{\pm}$ induce mappings $\Delta^{\pm}: \varepsilon^* \rightarrow \mathcal{V}^*$. Let $\omega \in \varepsilon^*$ and $f \in \mathcal{V}$, then

$$(\Delta^{\pm}\omega)(f) = \omega(\mathrm{D}^{\pm}(f)) \tag{11}$$

We have to prove that the mappings $\Delta^{\pm}$ may be restriced to pure states, i.e., that $\Delta^{\pm}$ map pure states into pure states. This restriction then defines the mappings $\partial^{\pm}: E \rightarrow V$ of Section 2.

It is easy to show from the conditions (8)–(10) that $\Delta^{\pm}$ map states into states. For the proof that pure states are mapped into pure states, we make use of the following characterization of a pure state in a commutative C^*-algebra:

If ω is a pure state over a commutative C^*-algebra $\mathcal{F}$, then the elements $f \in \mathcal{F}$ for which $\omega(f) = 0$ are a maximal ideal (which by definition is not equal to $\mathcal{F}$).

Now let $\omega \in V$ be a pure state and $f \in \varepsilon$ satisfy

$$(\Delta^{\pm}\omega)(f) = \omega(\mathrm{D}^{\pm}(f)) = 0$$

For any $g \in \varepsilon$ we therefore have

$$(\Delta^{\pm}\omega)(fg) = \omega(\mathrm{D}^{\pm}(fg)) = \omega(\mathrm{D}^{\pm}(f)\mathrm{D}^{\pm}(g)) = 0$$

The last equality follows since ω is a pure state and $\mathrm{D}^{\pm}f$ is an element of the maximal ideal satisfying $\omega(\mathrm{D}^{\pm}f) = 0$. Therefore, if ω is pure, so is $\Delta^{\pm}\omega$. This completes the reconstruction of the graph from its algebra.

Some of the requirements for a graph algebra, especially condition (8) and the requirements of the C^*-algebras having an identity element, may be replaced by weaker conditions, in which case the proof that pure states are mapped into pure states will become more elaborate.

5. COMMENTS

The presented purely algebraic formulation of a distance function on arbitrary graphs seems almost canonical. It does not depend on any further structures to be defined for a graph algebra. Furthermore, the algebraic settings as presented here easily translate to noncommutative cases, thereby defining the notion of a "noncommutative graph."

However, there is an obvious disadvantage of the construction presented above when compared to Connes' distance function on manifolds: The operator D is not self-adjoint. Furthermore, one would like to relate this operator to a kind of Dirac operator on graphs. Although the characterizing features of a Dirac operator on graphs are far from being obvious, one property should be that its square is related to the Laplace operator on graphs (see below).

The following ideas might sketch a way out of both problems. For simplicity we restrict ourselves to graphs without loops and multiple connections, although most of the ideas easily generalize to arbitrary graphs. Let us define a trace for $\mathcal{F}_V$ and $\mathcal{F}_E$ by

$$\mathrm{tr}_V(f) = \sum_{v \in V} f(v) \qquad \text{and} \qquad \mathrm{tr}_E(g) = \sum_{e \in E} g(e) \tag{12}$$

where $f \in \mathcal{F}_V$ and $g \in \mathcal{F}_E$. Algebraically, a trace may be defined as a linear functional from $\mathcal{F}_V$ or $\mathcal{F}_E$ into the complex numbers with the following properties (V/E means "V or E"):

$$\mathrm{tr}_{V/E}(f^*f) \geq 0$$
$$\mathrm{tr}_{V/E}(f_1 f_2) = \mathrm{tr}_{V/E}(f_2 f_1)$$
$$\mathrm{tr}_{V/E}(P(f)) = \mathrm{tr}_{V/E}(f) \qquad \text{for all} \quad P \in U(V/E)$$

where $U(V)$ and $U(E)$ denote the set of C^*-algebra isomorphisms of $\mathcal{F}_V$ and $\mathcal{F}_E$, respectively. For the commutative case of a graph algebra, $U(V)$ and $U(E)$ are given by the permutations of vertices and lines, respectively.

The trace operation allows the definiton of a scalar product:

$$(f, g) = \mathrm{tr}_{V/E}(fg) \tag{13}$$

(For nonfinite but countable vertex or edge sets one might have to restrict to square-summable functions, in which case the identity element is not part of the algebra. Some of the foregoing statements will be more difficult do prove, but these difficulties shall not concern us here.) Having a scalar product, we may define the adjoint operator D^T: $\mathcal{F}_E \to \mathcal{F}_V$ as usual. This adjoint operator is usually referred to as the incidence matrix (see, e.g., Biggs, 1974). We now consider the direct sum $\mathcal{G} = \mathcal{F}_V \oplus \mathcal{F}_E$ and define the operator

$$\hat{\mathrm{D}} = \begin{pmatrix} 0 & \mathrm{D}^T \\ \mathrm{D} & 0 \end{pmatrix} \tag{14}$$

It can be shown that:

1. $\hat{\mathrm{D}}$ is self-adjoint,
2. The operator

$$\hat{\mathrm{D}}^2 = \begin{pmatrix} \mathrm{D}^T\mathrm{D} & 0 \\ 0 & \mathrm{D}\mathrm{D}^T \end{pmatrix}$$

   has the interpretation of a Laplacian,
3. For ω_1 and ω_2 states over $\mathcal{F}_V$, the distance function

$$\hat{d}(\omega_1, \omega_2) = \sup\{|\omega_1(f) - \omega_2(f)|; \text{ for } f \in \mathcal{G} \text{ with } \|\hat{\mathrm{D}}f\| \leq 1\} \tag{15}$$

   is equivalent to the distance function given above [Eq. (7)].

While the first and last statement are easily proven, the second one requires perhaps a clarifying remark. First,

$$\mathrm{D}^T\mathrm{D} = V - A \qquad (\mathrm{D}^T\mathrm{D}: \quad \mathcal{F}_V \to \mathcal{F}_V)$$

where V is the valence matrix [for two vertices k, l we have $V_{kl} = v_k\delta_{kl}$, with v_k the valence (degree) of vertex k] and A the adjacency matrix. This matrix is known to be the discrete analog of the Laplacian on $\mathcal{F}_V$. Second,

$$\mathrm{DD}^T = 2 \cdot \mathrm{id}_E - A^L \qquad (\mathrm{DD}^T\colon \quad \mathcal{F}_E \to \mathcal{F}_E)$$

where A^L is a generalized adjacency matrix for a directed line graph (see, e.g., Biggs, 1974), defined by

$$(A^L)_{ef} = \begin{cases} 1 & e \text{ and } f \text{ share a common vertex, and are both} \\ & \text{incoming or outgoing} \\ -1 & e \text{ and } f \text{ share a common vertex, one is incoming,} \\ & \text{the other outgoing} \\ 0 & \text{otherwise} \end{cases}$$

This operator might be identified as the discrete analog of a Laplacian for (longitudinal) vector fields (compare, Filk, 1988). Hence, one necessary condition for the operator $\hat{\mathrm{D}}$ to be identified with a Dirac operator on a graph is fulfilled. The above results indicate that if a natural concept of a Dirac operator exists on arbitrary graphs (which is not at all self-evident), the two operators D and D^T will presumably be involved.

Two questions remain open: First, can the relation between $\hat{\mathrm{D}}$ and a Dirac operator be made more precise? The Dirac–Kähler formulation of fermions (Kaehler, 1962) might be of relevance here (see also Becher and Joos, 1982). Second, what is the interpretation of the distance functional (15) for pure states over $\mathcal{F}_E$?

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